

Quommutator deformations of $\text{spl}(N,1)$

Y. Brihaye

Department of Mathematical Physics

University of Mons

Av. Maistriau, B-7000 Mons, Belgium

February 9, 2008

Abstract

We analyse the Witten-Woronowicz's type deformations of the Lie superalgebras $\text{spl}(N,1)$ and obtain deformations parametrized by $N(N-1)/2$ independent parameters for $N > 2$ and by three parameters for $N = 2$. For some of these algebras, finite dimensional representations can be constructed in terms of finite difference operators, providing operators that are relevant for the classification of quasi exactly solvable, finite difference equations.

1 Introduction

The study of deformations of classical Lie algebras has attracted much attention in the last few years. Many types of deformations have been proposed. In some of them, the commutators of the generators are set equal to non linear (in fact transcendental) functions of some of the generators, they are the Drinfeld-Jimbo deformations [1, 2]. Other types of deformations consist in keeping the right hand sides of the structure relations linear in the generators while the commutators of the left hand side are deformed into quommutators, such deformations are called Witten-Woronowicz type [3, 4]. In this second approach, all the generators are treated on the same footing and the quommutator relations provide normal ordering rules which allow to write the elements of the enveloping algebra in a canonical form. This last property is useful for the kind of applications that we have in mind, namely the study of the quasi exactly solvable (QES), finite difference operators. The QES (differential or finite difference) equations refer to a class of spectral equations for which a finite part of the spectrum can be obtained by solving a system of algebraic equations [5, 6]. In this respect, the linear operators which preserve a finite dimensional vector space of smooth functions are naturally related to QES equations; such operators are also called quasi exactly solvable. The invariant space is typically a direct sum of modules of polynomials of given degrees.

The observations that QES operators possess hidden symmetries [5, 6] and that they can be related to some abstract algebras has motivated several tentatives to classify these operators and to study their properties in more

details. The scalar QES operators of one variable, preserving the vector space of polynomials of degree at most n , are related to the enveloping algebra of the Lie algebra $\mathfrak{sl}(2)$ [7]. Choosing, more generically, V as a direct sum of modules of polynomials, it is likely [8, 9, 10, 11, 12] that a finite number of basic QES operators can be put in correspondance with the generators of an abstract algebra, suitably represented in terms of differential operators. The more general QES operators can then be obtained as the polynomials in the basic operators, in other words they are the elements of an enveloping algebra.

The present understanding of QES equations suggests that the scalar QES equations are related to the classical Lie algebras [8, 9] while the systems of coupled QES equations are related to superalgebras [8, 10, 12]. For instance the superalgebra $\mathfrak{osp}(2,2)$ (or equivalently $\mathfrak{spl}(2,1)$) plays a crucial role [8, 10] for the systems of two equations of one variable. In the case of systems of two equations of V variables, the underlying symmetry is related to the superalgebras $\mathfrak{spl}(V+1,1)$ [12]

On the other hand it seems that, if we want to describe the algebraic structure underlying the QES finite difference equations, the representations of some deformed algebra emerge in a natural way. For instance, the scalar, finite difference QES operators in one variable [7] are related to the Witten deformation of $\mathfrak{sl}(2)$. Owing for this example and the results of [12] it is natural to try to classify the deformations of Witten-Woronowicz type of the Lie superalgebras $\mathfrak{spl}(N,1)$. This is the object of this paper; Drinfeld-Jimbo type deformations of general Lie superalgebras are discussed in [13]

2 The algebra $\text{spl}(N, 1)$

The graded Lie algebra $\text{spl}(N, 1)$ is generated by N^2 bosonic generators, N fermionic and N anti-fermionic generators. We will denote them respectively by E_a^b , V_a and \bar{V}^a , $a, b = 1, \dots, N$. The structure relations read as follows

$$\{V_a, V_b\} = 0 \quad , \quad (1)$$

$$\{\bar{V}^a, \bar{V}^b\} = 0 \quad , \quad (2)$$

$$\{V_a, \bar{V}^b\} = E_a^b \quad , \quad (3)$$

$$[E_a^b, V_c] = \delta_c^b V_a - \delta_a^b V_c \quad , \quad (4)$$

$$[E_a^b, \bar{V}^c] = \delta_a^b \bar{V}^c - \delta_a^c \bar{V}^b \quad , \quad (5)$$

$$[E_a^b, E_c^d] = \delta_c^b E_a^d - \delta_a^d E_c^b \quad . \quad (6)$$

As indicated in (6), the generators E_a^b form a $\text{gl}(N)$ subalgebra. Eqs.(4) (resp.(5)) show that the generators V_a 's (resp. \bar{V}^b) form an N -dimensional representation of $\text{gl}(N)$ under the adjoint action of the E_a^b .

We will consider the deformations of $\text{spl}(N, 1)$ obtained by replacing the commutators (4)-(6) and the anti-commutators (1)-(3) respectively by quommutators and anti-quommutators :

$$[A, B]_q = AB - qBA \quad , \quad \{A, B\}_q = AB + qBA \quad (7)$$

where q represents a parameter of deformation. More specifically, we classify the deformations obeying the following conditions:

- i) each commutator (resp. anti-quommutator) (1)-(6) defining the classical algebra is replaced by a quommutator (resp. anti-quommutator) (7) with its own parameter q .

ii) the couples of generators which (anti-) commute are imposed to (anti-) quommute.

iii) the right hand side of (3)-(5) is kept linear in the generators; in the right hand side of (6) we allow additional terms of the form $V_f \overline{V}^g$.

3 Deformations of $\text{spl}(N, 1)$

In order to present the deformations of $\text{spl}(N, 1)$ it is convenient to introduce a set of $\frac{N(N-1)}{2}$ arbitrary parameters labelled q_{ab} such that

$$q_{ba} = \frac{1}{q_{ab}} \quad , \quad a, b = 1, \dots, N \quad (8)$$

The deformations of $\text{spl}(N, 1)$ obeying the conditions above and compatible with the associativity conditions (i.e. the Jacobi identities) read as follows :

$$V_a V_b + q_{ab} V_b V_a = 0 \quad , \quad (9)$$

$$\overline{V}^a \overline{V}^b + q_{ab} \overline{V}^b \overline{V}^a = 0 \quad , \quad (10)$$

$$V_a \overline{V}^b + q_{ba} \overline{V}^b V_a = E_a^b \quad (11)$$

$$E_a^b V_c - q_{ac} q_{cb} V_c E_a^b = \delta_c^b V_a - \delta_a^b V_c \quad , \quad (12)$$

$$E_a^b \overline{V}^c - q_{bc} q_{ca} \overline{V}^c E_a^b = \delta_a^b \overline{V}^c - q_{ba} \delta_a^c \overline{V}_b \quad , \quad (13)$$

$$E_a^b E_c^d - q_{ac} q_{cb} q_{bd} q_{da} E_c^d E_a^b = \delta_b^c E_a^d - q_{ba} q_{ac} q_{cb} \delta_a^d E_c^b \quad . \quad (14)$$

The normalisation of the generators has been choosen in such a way that the right hand sides of (11),(12) look as simple as possible. The bosonic generators close under the quommutators (14), leading to a deformation of

the Lie algebra $\mathfrak{gl}(N)$. It can be checked that the products of three and of four q 's defining the structure constants in (14) depend effectively on $\frac{(N-1)(N-2)}{2}$ parameters among the whole set of the $\frac{N(N-1)}{2}$ parameters q_{ab} . In fact, the algebra (14) coincides with the deformation of $\mathfrak{gl}(N)$ constructed in [15]. It is amusing to note that the parameters q_{ab} , used to label the deformations in [15], appear here as defining the anti-quommutators (9),(10) of the fermionic operators. Note, however, that another quommutator is used in [15], it can be related to (14) by an appropriate renormalisation of the generators.

4 Deformations of $\mathfrak{spl}(2, 1)$

The analysis of Ref.[15] reveals that the algebra $\mathfrak{gl}(2)$ leads to a richer set of deformations than for the generic case $\mathfrak{gl}(N)$ for $N > 2$. A similar result holds for the deformations of $\mathfrak{spl}(N, 1)$. We now discuss the deformations of $\mathfrak{spl}(2, 1)$ which, remember, is equivalent to the (perhaps more popular) graded Lie algebra $\mathfrak{osp}(2, 2)$.

The most general deformation of $\mathfrak{spl}(2, 1)$ obeying the three restrictions above and compatible with associativity depends on three parameters, say p, r, s . The different relations read as follows

- for the fermionic-fermionic relations

$$V_1 V_1 = V_2 V_2 = 0 \quad , \quad \overline{V}^1 \overline{V}^1 = \overline{V}^2 \overline{V}^2 = 0 \quad (15)$$

$$\{V_1, V_2\}_{\frac{p}{sr}} = 0 \quad , \quad \{\overline{V}^1, \overline{V}^2\}_{\frac{s}{pr}} = 0 \quad (16)$$

$$\{\overline{V}^j, V_j\} = E_j^j \quad , \quad j = 1, 2 \quad (17)$$

$$\{\bar{V}^1, V_2\}_{psr} = E_2^1 \quad , \quad \{\bar{V}^2, V_1\}_{\frac{ps}{r}} = E_1^2 \quad (18)$$

- for the fermionic-bosonic relations

$$\begin{aligned} [E_1^1, V_1] &= 0, & [E_2^2, V_1]_{s^2} &= V_1 \\ [E_1^2, V_1]_{ps/r} &= 0, & [E_2^1, V_1]_{sr/p} &= -\frac{sr}{p}V_2 \\ [E_1^1, V_2]_{p^2} &= V_2, & [E_2^2, V_2] &= 0 \\ [E_1^2, V_2]_{p/sr} &= -\frac{p}{sr}V_1, & [E_2^1, V_2]_{psr} &= 0 \end{aligned} \quad (19)$$

$$\begin{aligned} [E_1^1, \bar{V}^1] &= 0, & [E_2^2, \bar{V}^1]_{1/s^2} &= -\frac{1}{s^2}\bar{V}^1 \\ [E_1^2, \bar{V}^1]_{ps/r} &= \bar{V}^2, & [E_2^2, \bar{V}^1]_{1/psr} &= 0 \\ [E_1^1, \bar{V}^2]_{1/p^2} &= -\frac{1}{p^2}\bar{V}^2, & [E_2^2, \bar{V}^2] &= 0 \\ [E_1^2, \bar{V}^2]_{r/ps} &= 0, & [E_2^1, \bar{V}^2]_{s/pr} &= \bar{V}^1 \end{aligned} \quad (20)$$

- for the bosonic-bosonic operators

$$[E_1^1, E_2^2] = 0 \quad , \quad (21)$$

$$[E_1^1, E_1^2]_{1/p^2} = -\frac{1}{p^2}E_1^2, \quad [E_2^2, E_1^2]_{s^2} = E_1^2 \quad (22)$$

$$[E_1^1, E_2^1]_{p^2} = E_2^1, \quad [E_1^1, E_2^1]_{1/s^2} = -\frac{1}{s^2}E_2^1 \quad (23)$$

$$[E_1^2, E_2^1]_{s^2/p^2} = E_1^1 - \frac{s^2}{p^2}E_2^2 + (s^2 - 1)V_1\bar{V}^1 - \frac{s^2}{p^2}(p^2 - 1)V_2\bar{V}^2 \quad (24)$$

The three parameters of the deformation, noted p, r, s , are intrinsic and cannot be eliminated by a rescaling of the generators. This is seen easily from the way these parameters enter in the different coefficients defining the quommutator. The undeformed $\text{osp}(2,2)$ algebra is recovered by $p = s = r = 1$.

In the limit $s^2 = p^2 = 1$, the generic case of the previous section is recovered : there is only one deforming parameter, noted r . The $\mathfrak{gl}(2)$ subalgebra is undeformed.

Keeping r, s, p arbitrary and decoupling the fermionic generators from eq.(24) leads, together with (21), (22) and (23), to the two parameters deformation of $\mathfrak{gl}(2)$ obtained in ref. [14] (a four parameters deformation of $\mathfrak{gl}(2)$ was further obtained in ref.[15]).

5 Representations

Restricting the three parameters p, r, s in the algebra presented in sect. 4 to the case

$$p = s^{-1} = q^{\frac{1}{2}} \quad , \quad r = 1 \quad (25)$$

leads to a one parameter deformation of $\mathfrak{osp}(2,2)$ parametrized by q which we will refer to as $\mathfrak{osp}(2,2)_q$. It is equivalent, up to a rescaling of the generators, to the deformation first discussed in [16]. As for the Witten type deformation of $\mathfrak{sl}(2)$ [7], it is possible to construct some representations of $\mathfrak{osp}(2,2)_q$ in terms of a finite difference operator D_q :

$$D_q f(x) = \frac{f(x) - f(qx)}{(1-q)x} \quad , \quad D_q x^n = [n]_q x^{n-1} \quad , \quad [n]_q \equiv \frac{1-q^n}{1-q} \quad (26)$$

The representations we want to discuss act on the vector space $P(n-1) \oplus P(n)$ (where $P(n)$ denotes the module of polynomials of degree n in the variable x). It is therefore of dimension $2n-1$. The fermionic generators are represented by

$$V_1 = \sigma_- , \quad V_2 = x\sigma_-$$

$$\bar{V}_1 = q^{-n}(xD_q - [n]_q)\sigma_+, \quad \bar{V}_2 = q^{-1}D_q\sigma_+ \quad (27)$$

where $\sigma_{\pm} = (\sigma_1 \pm i\sigma_2)/2$. The operators E_j^i are easily constructed with (16),(17). The 2×2 matrix, finite difference operators preserving $P(n-1) \oplus P(n)$ are then the elements of the enveloping algebra generated by (27). By construction, these operators are quasi exactly solvable.

6 Deformations of $\text{osp}(1,2)$

Recently a Witten type deformation of the super Lie algebra $\text{osp}(1,2)$ was constructed [17]. We showed that this deformation is the only one to fulfil the three requirements of sect. 2, its finite dimensional representations can be expressed in terms of the operator (26). Again the space of the representation is $P(n-1) \oplus P(n)$ and the two fermionic generators of $\text{osp}(1,2)$ are represented by 2×2 matrix operators. Using exactly the same notation as in ref.[17] we find

$$V_- = \begin{pmatrix} 0 & D_{q^2} \\ 1 & 0 \end{pmatrix}, \quad V_+ = \begin{pmatrix} 0 & q^{-2n}(xD_{q^2} - [n]_{q^2}) \\ x & 0 \end{pmatrix} \quad (28)$$

The three bosonic operators, H, J_-, J_+ can then be computed through the structure of $\text{osp}(1,2)$, namely

$$H = \{V_-, V_+\}_q, \quad J_- = \{V_-, V_-\}_q, \quad J_+ = \{V_+, V_+\}_q \quad (29)$$

Correspondingly, the Casimir operator (eq.(13) in ref. [17]) has a value $C = -\frac{1}{2}[-n - \frac{1}{2}]_{q^2}$

7 Concluding remarks

The representations of $\text{spl}(V+1,1)$ formulated in terms of differential operators provide the building blocks for the construction of the QES operators preserving more general polynomial spaces of V variables [12]. This result, and the fact that the quommutators deformations of super Lie algebras have not been studied systematically (at least to our knowledge), motivate a study of the deformations of the Lie superalgebras $\text{spl}(N,1)$ by means of quommutators.

On the other hand, the most interesting examples of QES systems (e.g. the relativistic Coulomb problem [19], the doubly periodic Lamé equation and the stability of the sphaleron in the abelian Higgs model [10]) are related to the algebra $\text{osp}(2,2)$. Generalisations of this algebra, e.g. its deformations, and the corresponding realizations in terms of finite difference operators therefore deserve some attention. The operators (27) could be relevant for the study of some coupled problems arising in discrete quantum mechanics in [18].

On a more abstract level we mention that the operators (27) can be used to construct finite difference operators preserving the vector space $P(m) \oplus P(n)$. The algebraic structure underlying these operators is probably determined by a series of deformed, non-linear superalgebras indexed by $|m-n|$. Finally, it would be interesting to relate the two parameters deformation of $\text{osp}(2,2)$ of ref.[20] with our deformation. The occurrence of some mapping between the two structures would allow to transport the Hopf structure of to our case.

References

- [1] V. Drinfeld, Proc.Intern.Congress of Mathematicians (Berkeley, 1986) 798.
- [2] M. Jimbo, Lett. Math. Phys. 10 (1985) 63.
- [3] E. Witten, Nucl. Phys. B **330**, 285 (1990).
- [4] S. Woronowicz, Commun. Math. Phys. **111** 613 (1987).
- [5] A.V. Turbiner, Comm. Math. Phys. **119**, 467 (1988).
- [6] A.G. Ushveridze, *Quasi exact solvability in quantum mechanics* (Institute of Physics Publishing, Bristol and Philadelphia, 1993).
- [7] A.V. Turbiner, J. Phys. A **25**, L 1087 (1992).
- [8] M.A. Shifman and A.V. Turbiner, Comm. Math. Phys. **120**, 347 (1989).
- [9] A. Gonzalez-Lopez, N. Kamran, P.J. Olver, J.Phys. A **24** 3995 (1991).
- [10] Y. Brihaye, P. Kosinski, J. Math. Phys.**35**, 3089 (1994).
- [11] Y. Brihaye, S. Giller, C. Goneru, P. Kosinski, J. Math. Phys.**36**, 4340 (1995)
- [12] Y.Brihaye, J.Nuyts, *The hidden symmetry algebras of a class of quasi exactly solvable multi dimensional equations*, q-alg:9701016.
- [13] R. Floreanini, V. Spiridonov, and L. Vinet, Commun. Math. Phys. **137** 149 (1991).

- [14] D.B. Fairlie and C.K. Zachos, Phys.Lett.B **256** 43 (1991).
- [15] D.B. Fairlie, J. Nuyts, J. Math.. Phys.**35**,3794 (1994).
- [16] Y. Brihaye, S. Giller, P. Kosinski, *On the relations between $osp(2,2)$ and the quasi exactly solvable systems*, hep-th:9611226.
- [17] W. Chung, *On Witten-type deformation of $osp(1/2)$ algebra*, q-alg:9609015.
- [18] P.B. Wiegmann, A.V. Zabrodin, Nucl. Phys. B **451**, 699 (1995).
- [19] Y.Brihaye, N.Devaux, P. Kosinski, Int. J. Mod. Phys. A **10** (1995) 4633.
- [20] P. Parashar, J. Phys. A **27** (1994) 3803.